LECTURE 10 :

VECTOR-VALUED FUNCTIONS

SO FAR, WE'VE STUDIED INTERPOLATION & EXTENSION PROBLEMS FOR SCALAR-VALUED FNS.

$$f : E \to \mathbb{R}.$$

TODAY, WE LOOK AT THE ANALOGOUS PROBLEMS FOR VECTOR-VALUED FNS.

$$f : E \longrightarrow Y$$

$$\left\{ \begin{array}{l} Y = \mathbb{R}^D, \text{ OR MAYBE} \\ \qquad \text{A BANACH SPACE} \end{array} \right\}$$

We discuss problems re

VECTOR-VALUED FNS.

in order of increasing difficulty:

Formal consequences of previous results for scalar f

Results that reduce to the scalar case by a trick

RESULTS ON VECTOR-VALUED FNS.

THAT REQUIRE SUBSTANTIAL NEW IDEAS.

UNSOLVED PROBLEMS.

SETUP :

Let Y be a Banach space,

with norm $|\cdot|_Y$.

Let Y^* be its dual space,

with norm $|\cdot|_{Y^*}$.

$C^m(\mathbb{R}^n, Y)$ DENOTES THE SPACE

OF FUNCTIONS

$$F : \mathbb{R}^n \to Y$$

WITH CONTINUOUS & BOUNDED

DERIVATIVES up to

ORDER m.

The norm on $C^m(\mathbb{R}^n, Y)$ is given by

$$\boxed{\|F\|_{C^m(\mathbb{R}^n, Y)} = \sup_{x \in \mathbb{R}^n} \max_{|\alpha| \le m} |\partial^\alpha F(x)|_Y}$$

FOR ANY $E \subset \mathbb{R}^n$,

$\boxed{C^m(E, Y) \text{ DENOTES THE SPACE OF ALL } F|_E \text{ FOR } F \in C^m(\mathbb{R}^n, Y).}$

$\left\{ \begin{array}{l} E \text{ finite} \Rightarrow C^m(E, Y) \\ \text{CONSISTS OF ALL FNS.} \\ f : E \to Y. \end{array} \right\}$

The norm on $C^m(E, Y)$

is given by

$$\boxed{\|f\|_{C^m(E,Y)} = \inf\left\{ \|F\|_{C^m(\mathbb{R}^n, Y)} : \begin{array}{l} F \in C^m(\mathbb{R}^n, Y) \\ F = f \text{ on } E \end{array} \right\}.}$$

CONSTANTS DENOTED by

$c, C, C', \ldots$

WILL ALWAYS DEPEND

only on m, n in

$C^m(\mathbb{R}^n, Y)$,

NEVER ON f, E or Y.

Let $f \in C^m(E, Y)$, $F \in C^m(\mathbb{R}^n; Y)$.

Then F is a <u>C-optimal extension</u> of f

$\Longleftrightarrow$

$$\boxed{\begin{gathered} F = f \text{ on } E \\ \text{and} \\ \|F\|_{C^m(\mathbb{R}^n;\, Y)} \le C\, \|f\|_{C^m(E,\, Y)}. \end{gathered}}$$

FORMAL CONSEQUENCES of RESULTS FOR SCALAR FNS.

REFINED FINITENESS THM :

Suppose E is FINITE, $\#(E) = N$.

Then, in at most

$C N \log N$ computer ops.,

WE can compute SUBSETS

$S_1, S_2, \ldots, S_L \subset E$

such that ...

$$\boxed{\#(S_\ell) \leq C \text{ for each } \ell = 1, \dots, L}$$

$$\boxed{L \leq CN}$$

For any $f \in C^m(E, Y)$, we have

$$\| f \|_{C^m(E,Y)} \leq C \max_\ell \| (f|_{S_\ell}) \|_{C^m(S_\ell, Y)}$$

[SAME $S_1, \dots, S_L$ as in SCALAR CASE!]

LINEAR EXTENSION OPERATOR OF BOUNDED DEPTH.

Thm: Let $E \subset \mathbb{R}^n$ finite,

$\#(E) = N$.

Then there exists a linear map

$T : C^m(E, Y) \to C^m(\mathbb{R}^n; Y)$

such that ...

$$\boxed{\begin{array}{l} Tf = f \text{ on } E \\ \quad \text{for all } f \in C^m(E, Y) \end{array}}$$

$$\boxed{\begin{array}{l} \|Tf\|_{C^m(\mathbb{R}^n; Y)} \leq C \|f\|_{C^m(E, Y)} \\ \text{for all } f \in C^m(E, Y). \end{array}}$$

Moreover, ...

T is given by a FORMULA:

$$\boxed{Tf(x) = \sum_{y \in S(x)} \lambda(x,y) f(y)}$$

for each $x \in \mathbb{R}^n$,

where $\boxed{S(x) \subset E}$,

$\boxed{\lambda(x,y),\ S(x) \text{ are independent of } f,}$

and

$\boxed{\#(S(x)) \leq C \text{ for each } x \in \mathbb{R}^n}$

Finally, the $\lambda(x, y)$ and $S(x)$ may be efficiently computed.

[SAME $\lambda(x, y)$, $S(x)$
AS IN THE SCALAR CASE!]

These results give rise to efficient computation of C-optimal extensions, as in the SCALAR CASE.

The above Thms follow from the corresponding results for $f : E \to \mathbb{R}$, simply because

$$\boxed{|v|_Y = \sup\{|\langle v^*, v\rangle| : v^* \in Y^*, |v^*|_{Y^*} = 1\}}$$

RESULTS THAT REDUCE TO THE SCALAR CASE BY A TRICK.

WE DISCUSS RESULTS

for $C^m(E, Y)$

when $E \subset \mathbb{R}^n$ IS INFINITE.

WE GENERALIZE

BUNDLES, GLAESER REF'S,

etc. to the case of

VECTOR-VALUED FNS.

WHY SHOULD

WE

CARE ?

THE BRENNER-EPSTEIN-HOCHSTER-KOLLÁR PROBLEM :

Let $\left(A_{ij}(x)\right)_{\substack{i=1,\dots,I \\ j=1,\dots,J}}$ be a matrix of poly's on $\mathbb{R}^n$.

Let $f_1, \dots, f_I$ be polys on $\mathbb{R}^n$.

WE CONSIDER THE

SYSTEM OF EQUATIONS

$$\boxed{\sum_{j=1}^{J} A_{ij}(x) F_j(x) = f_i(x) \qquad (i=1,\dots,I)} \leftarrow (!)$$

for unknown functions

$F_1, \dots, F_J$ on $\mathbb{R}^n$.

WE ASK WHETHER (!)

ADMITS A SOLUTION

with $F_1, \dots, F_J$ <u>CONTINUOUS</u>.

An Example (due to M. Hochster)

Consider the equation

$$x^2 F + y^2 G + xyz^2 H = f(x,y,z)$$

for unknown continuous fns. $F(x,y,z)$, $G(x,y,z)$, $H(x,y,z)$.

$\boxed{\text{Which poly's } f \text{ admit a continuous solution?}}$

VECTOR-VALUED GENERALIZATIONS OF OUR RESULTS ON BUNDLES, GLAESER REF'S, ...

APPLY TO THE Brenner-Hochster-Epstein-Kollár Prob., WITH "CONTINUOUS" replaced by "C^m".

Moreover, the required generalizations of our results re BUNDLES, ... follow easily from the scalar case by a TRICK.

We first set up the vector-valued versions of bundles & Glaeser refinements, next explain the TRICK, and then give applications to the BRENNER-EPSTEIN-HOCHSTER-KOLLÁR PROBLEM.

Setup:

Fix $m, n, D \geq 1$.

We work in $C^m(\mathbb{R}^n, \mathbb{R}^D)$.

If $F = (F_1, \dots, F_D) \in C^m(\mathbb{R}^n, \mathbb{R}^D)$,

then the jet of F at x

is defined in the obvious way:

$J_x(F) = (J_x(F_1), \dots, J_x(F_D))$,

where $J_x(F_j)$ is the

DEGREE m (NOT $m-1$)

TAYLOR POLY OF F_j at x.

Thus, $J_x(F)$ belongs to

$\mathcal{P}(\mathbb{R}^n, \mathbb{R}^D)$,

the vector space of

$\mathbb{R}^D$-valued poly's on $\mathbb{R}^n$

of degree $\leq m$.

For fixed $x \in \mathbb{R}^n$,

jet multiplication

$$P \odot_x Q \equiv J_x(PQ)$$

makes $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^1)$

into a ring $\mathcal{R}_x$,

the <u>ring of jets at x.</u>

For $P \in \mathcal{R}_x$ and

$Q = (Q_1, \dots, Q_D) \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^D)$,

WE MAKE THE OBVIOUS DEF.

$$\boxed{P \odot_x Q = (P \odot_x Q_1, \dots, P \odot_x Q_D) \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^D).}$$

This makes $\mathcal{P}(\mathbb{R}^n, \mathbb{R}^D)$

into an $\mathcal{R}_x$-MODULE,

which we call $\mathcal{R}_x^D$.

Let $E \subset \mathbb{R}^n$ be COMPACT.

A <u>BUNDLE</u> OVER E IS A FAMILY

$\mathcal{H} = (H(x))_{x \in E}$, where,

for each $x \in E$,

EITHER $H(x)$ IS <u>EMPTY</u>

or

$$\boxed{\begin{array}{l} H(x) = f(x) + I(x), \ \& \\ f(x) \in \mathcal{R}_x^D, \\ I(x) \subset \mathcal{R}_x^D \text{ is an } \mathcal{R}_x\text{-SUBMODULE.} \end{array}}$$

As in the SCALAR CASE,

we call $H(x_0)$

the **FIBER** of

$$\mathcal{H} = \left(H(x) \right)_{x \in E}$$

at x_0.

When we want to recall which m, n, D we are using, we speak of a "BUNDLE with respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$".

When $D=1$, we speak of a "BUNDLE with respect to $C^m(\mathbb{R}^n)$" or SIMPLY A "SCALAR BUNDLE"

A VECTOR-VALUED FUNCTION

$F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$

is a <u>SECTION</u> of

$\mathcal{H} = (H(x))_{x \in E}$

if

$J_x(F) \in H(x)$ for all $x \in E$.

QUESTION:

DECIDE WHETHER A GIVEN BUNDLE $\mathcal{H}$ HAS A SECTION.

AS IN THE SCALAR CASE, WE ANSWER THIS QUESTION WITH GLAESER REFINEMENTS.

Let $\mathcal{H} = (H(x))_{x \in E}$

be a Bundle.

The <u>Glaeser Refinement</u>

of $\mathcal{H}$ is the bundle

$\tilde{\mathcal{H}} \equiv (\tilde{H}(x))_{x \in E}$,

where, for each $x_0 \in E$,

$\tilde{H}(x_0)$ is defined as follows:

Fix a large enough $k = k(m, n, D)$.

Then $\tilde{H}(x_0)$ consists of all $P_0 \in H(x_0)$ such that:

Given $\varepsilon > 0$ there exists $\delta > 0$ s.t. for all $x_1, \dots, x_k \in E \cap B(x_0, \delta)$, there exist $P_1 \in H(x_1), \dots, P_k \in H(x_k)$ s.t.

$$|\partial^\alpha (P_i - P_j)(x_i)| \leq \varepsilon |x_i - x_j|^{m-|\alpha|}$$

for $0 \leq i, j \leq k$, $|\alpha| \leq m$.

Except that

$$\boxed{\begin{array}{l} P_1, \dots, P_k \text{ are } \mathbb{R}^D\text{-valued} \\ \text{and} \\ k = k(m,n,D) \text{ rather than } k(m,n) \end{array}}$$

this is the same definition as in the SCALAR CASE.

MINOR REMARK:

As in the scalar case,

$|x_i - x_j|^{m-|\alpha|}$ becomes 0^0

if $x_i = x_j$ and $|\alpha| = m$.

In this degenerate case,

we declare that

$|x_i - x_j|^{m-|\alpha|} = 0$.

ITERATED GLAESER REFINEMENT

Let $\mathcal{H}$ be a bundle.

The <u>iterated Glaeser refinements</u> of $\mathcal{H}$ are the bundles $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \dots$, where

$$\boxed{\begin{aligned} &\mathcal{H}_0 = \mathcal{H} \quad \text{and} \\ &\mathcal{H}_{\ell+1} = \text{Glaeser refinement of } \mathcal{H}_\ell \end{aligned}}$$

GLAESER'S LEMMA:

For a large enough $l_* = l_*(m,n,D)$ we have

$$\mathcal{H}_{l_*} = \mathcal{H}_{l_*+1} = \mathcal{H}_{l_*+2} = \cdots$$

(SAME PROOF AS IN THE SCALAR CASE)

WE CALL $\mathcal{H}_{l_*}$ THE STABLE GLAESER REFINEMENT of $\mathcal{H}$.

AS IN THE SCALAR CASE:

VECTOR-VALUED MAIN THM:

Let $\mathcal{H}$ be a bundle,

and let $\mathcal{H}_{\ell_*}$ be its

stable Glaeser refinement.

Then $\mathcal{H}$ has a section

if & only if

$\mathcal{H}_{\ell_*}$ has NO EMPTY FIBERS.

As promised, the

VECTOR-VALUED MAIN THM.

follows from the

KNOWN SCALAR CASE

by a SIMPLE TRICK,

WHICH WE NOW EXPLAIN.

We will work in $\mathbb{R}^n$, and in $\mathbb{R}^{n+D}$.

We denote points of $\mathbb{R}^n$ by

$$x = (x_1, \dots, x_n),$$

We denote points of $\mathbb{R}^D$ by

$$\xi = (\xi_1, \dots, \xi_D),$$

and we denote points of $\mathbb{R}^{n+D}$ by

$$(x, \xi) = (x_1, \dots, x_n, \xi_1, \dots, \xi_D).$$

Let $P(x,\xi)$ be a (SCALAR-VALUED) polynomial on $\mathbb{R}^{n+D}$.

Then $\nabla_\xi P(x,\xi)$ denotes the $\mathbb{R}^D$-VALUED poly.

$$\left(\frac{\partial}{\partial \xi_1} P, \ldots, \frac{\partial}{\partial \xi_D} P\right)$$

on $\mathbb{R}^{n+D}$.

Let $\mathcal{H} = (H(x))_{x \in E}$ be a bundle with respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$.

$\boxed{\text{We will define a bundle } \mathcal{H}^+ = (H^+(x,\xi))_{(x,\xi) \in E^+} \text{ with respect to } C^{m+1}(\mathbb{R}^{n+D}, \mathbb{R}^1) \text{ as follows:}}$

BECAUSE $\mathcal{H}^+$ IS A BUNDLE

WITH RESPECT TO

$C^{m+1}(\mathbb{R}^{n+D}, \mathbb{R}^1)$,

EACH FIBER $H^+(x, \xi)$

WILL SIT INSIDE $\mathcal{P}^+$,

THE VECTOR SPACE OF ALL (SCALAR VALUED)

POLYS ON $\mathbb{R}^{n+D}$ OF DEGREE

AT MOST $m+1$

We set

$$\boxed{E^+ = E \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}^D.}$$

For $(x^0, 0) \in E^+$, we take

$H^+(x^0,0)$ to consist of all

$P \in \mathcal{P}^+$ such that

$$\boxed{P(x,0) = 0 \quad (\text{all } x) \quad \text{and} \quad \nabla_\xi P(x,0) \in H(x^0)}$$

$$\boxed{\text{That defines } \mathcal{H}^+ = \left(H^+(x,\xi)\right)_{(x,\xi)\in E^+}.}$$

One checks easily that $\mathcal{H}^+$ is a bundle with respect to $C^{m+1}(\mathbb{R}^{n+D}, \mathbb{R}^1)$.

The MAIN THM for the (VECTOR-VALUED) bundle $\mathcal{H}$ follows <u>easily</u> from the (KNOWN!) MAIN THM for the (SCALAR) bundle $\mathcal{H}^+$, together with the classical Whitney extension theorem.

DETAILS

OMITTED

HERE —

LET'S JUST

BELIEVE IT.

P.S. This trick is

due to cf &

KEVIN LULI.

This concludes our explanation of the SIMPLE TRICK.

Now let's return to the BRENNER-EPSTEIN-HOCHSTER-KOLLÁR PROBLEM.

WE START WITH A SIMPLE REMARK ON OUR NOTION OF $C^m(\mathbb{R}^n)$.

So far, we've defined

$C^m(\mathbb{R}^n)$ to consist of

all $F : \mathbb{R}^n \to \mathbb{R}$

whose derivatives

up to order m

are continuous

and bounded

on $\mathbb{R}^n$.

WE NOW CHANGE OUR NOTATION:

$C^m(\mathbb{R}^n)$ CONSISTS OF ALL $F : \mathbb{R}^n \to \mathbb{R}$ WHOSE DERIVATIVES UP TO ORDER m ARE CONTINUOUS.

(No Growth Conditions at ∞!)

WE WILL <u>IGNORE</u> ANY DIFFICULTIES ARISING FROM OUR PASSING FROM ONE NOTION OF C^m TO THE OTHER.

(Those difficulties are MINOR – trust me!)

Now let's introduce our version of the Brenner-Epstein-Hochster-Kollár Problem.

For simplicity, we work here with a single linear equation for D unknown functions $F_1, \dots, F_D$, although our results apply equally well to systems of equations.

FIX FUNCTIONS

$A_1(x), \cdots, A_D(x)$ on $\mathbb{R}^n$.

WE WILL STUDY THE EQUATION

$$\boxed{A_1(x)F_1(x) + \cdots + A_D(x)F_D(x) = f(x)}$$

for UNKNOWN FUNCTIONS

$F_1, \cdots, F_D \in C^m(\mathbb{R}^n)$

for $\boxed{\text{FIXED } m \geq 0.}$

BECAUSE m IS FIXED,

WE'RE NOT ALLOWED

TO LOSE DERIVATIVES.

WE ASK THREE QUESTIONS

Q1: Fix $A_1(x), \dots, A_D(x), f(x)$ ARBITRARY FUNCTIONS.

How can we decide whether there exist $F_1, \dots, F_D \in C^m(\mathbb{R}^n)$ s.t.
$A_1 F_1 + \dots + A_D F_D = f$ on $\mathbb{R}^n$?

Q2 : Fix <u>POLYNOMIALS</u> $A_1, \dots, A_D$.

The <u>polynomials</u> f for which

$$A_1 F_1 + \dots + A_D F_D = f$$

admits a C^m solution $F_1, \dots, F_D$

form an <u>IDEAL</u> in the ring

of polynomials.

PROBLEM: EXHIBIT GENERATORS

Q3: Let $A_1, \dots, A_D, f$ be <u>polys.</u> on $\mathbb{R}^n$.

Suppose the equation

$$A_1 F_1 + \dots + A_D F_D = f$$

has a C^m solution $F_1, \dots, F_D$.

$\boxed{\text{Can we take } F_1, \dots, F_D \text{ to be } C^m \text{ SEMIALGEBRAIC FUNCTIONS?}}$

(WILL RECALL THE DEF. SOON)

OUR VECTOR-VALUED

MAIN THM ON BUNDLES

ALLOWS US TO ANSWER

Q1

COMPLETELY.

BUILDING ON THE

VECTOR-VALUED

MAIN THM FOR BUNDLES,

WE CAN GIVE A

COMPLETE ANSWER TO

Q2.

WE DON'T KNOW

THE ANSWER TO

Q3.

SEMIALGEBRAIC SETS & FUNCTIONS

A SUBSET $E \subset \mathbb{R}^n$ IS SEMIALGEBRAIC IF IT CAN BE EXPRESSED AS A UNION OF FINITELY MANY SETS of the form

$$\{x \in \mathbb{R}^n : P_\ell(x) = 0 \quad (\ell = 1, \cdots, L) \ \& \ Q_\ell(x) > 0 \quad (\ell = 1, \cdots, L')\}$$

Here, the P_ℓ & Q_ℓ are polynomials on $\mathbb{R}^n$.

We allow the case $L=0$ (no P_ℓ's) and/or $L'=0$ (no Q_ℓ's).

Let $\varphi : E \to \mathbb{R}^k$ be a function defined on some $E \subset \mathbb{R}^n$.

Then φ is a

<u>SEMIALGEBRAIC FUNCTION</u>

if its GRAPH

$\{(x, \varphi(x)) : x \in E\} \subset \mathbb{R}^{n+k}$

is a SEMIALGEBRAIC SET.

SIMPLE EXAMPLES of SEMIALG. SETS:

$$\boxed{\{(x_1, x_2, x_3, t) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 = t^2,\ t \geq 0\}}$$

$$\boxed{\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1,\ x_1 > 0\}}$$

$$\boxed{\{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_1^2 - x_2^3 = 0\}}$$

Simple Examples of Semialg. Functions:

$$\varphi : \mathbb{R} \to \mathbb{R}$$

$$\varphi(x) = \begin{cases} \sqrt{1-x^2} & \text{if } -1 \leq x \leq 1 \\ 17 & \text{otherwise} \end{cases}$$

$$\psi : \mathbb{R}^2 \to \mathbb{R}^3$$

$$\psi(x,y) = \begin{cases} \left(x+y,\ 17,\ \sqrt{x^2+y^2}\right) & \text{if } y>2 \\ (-y,\ +x,\ 0) & \text{otherwise} \end{cases}$$

SOLUTION TO Q1

Let $A_1, \dots, A_D, f$ be ARBITRARY given fns. on $\mathbb{R}^n$.

We ask whether the EQUATION

$$A_1 F_1 + \dots + A_D F_D = f$$

admits a C^m solution.

To answer this, we introduce

a bundle

$\mathcal{H} = (H(x))_{x \in \mathbb{R}^n}$

with respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$

as follows.

(Oops! $\mathbb{R}^n$ ISN'T COMPACT.

NEVER MIND ...)

DEFINING THE BUNDLE

$\mathcal{H} = (H(x))_{x \in \mathbb{R}^n}$

For each $x^0 \in \mathbb{R}^n$, $H(x^0)$ consists

of all $(P_1, \dots, P_D) \in \mathcal{P}(\mathbb{R}^n, \mathbb{R}^D)$

such that

$$A_1(x_0) P_1(x_0) + \dots + A_D(x_0) P_D(x_0) = f(x_0).$$

Then

$\boxed{\mathcal{H} \text{ is a bundle}}$

and

A section of $\mathcal{H}$ is precisely a D-tuple of functions $F_1, \dots, F_D \in C^m(\mathbb{R}^n)$ such that $A_1 F_1 + \dots + A_D F_D = f$ on $\mathbb{R}^n$.

So our EQUATION

$$A_1 F_1 + \cdots + A_D F_D = f$$

has a C^m SOLUTION

$$\Longleftrightarrow$$

The bundle $\mathcal{H}$ has a section.

To decide whether $\mathcal{H}$ has a section, we compute its stable Glaeser refinement $\mathcal{H}_{l_*}$, and check whether all the fibers of $\mathcal{H}_{l_*}$ are NON-EMPTY.

That's the Solution to Q1.

WHAT ABOUT Q2?

Given polys $A_1, \cdots, A_D$,

we want to find generators

for the ideal of <u>polys</u> f s.t.

the EQUATION

$$A_1 F_1 + \cdots + A_D F_D = f$$

admits a C^m solution.

This problem can't be solved by ANALYSIS alone, because it asks for generators of an ideal in a polynomial ring.

It can't be solved by ALGEBRA alone, because it concerns C^m functions.

SO WE SPLIT THE PROBLEM
INTO A PROBLEM INVOLVING
ONLY ANALYSIS
and A PROBLEM INVOLVING
ONLY ALGEBRA.

Here's the Analysis Question:

Given polys $A_1, \cdots, A_D$, characterize the C^∞ functions f such that the equation

$$A_1 F_1 + \cdots + A_D F_D = f$$

admits a C^m solution for fixed m.

AN EXAMPLE

LET'S RETURN TO

HOCHSTER'S EQUATION

$$x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f.$$

Suppose $f \in C^\infty$.

Then a continuous solution exists $\iff$

$$\boxed{\begin{array}{l} f = \partial_x f = \partial_y f = 0 \\ \text{for } x = y = 0 \text{ (all } z) \\ \\ \text{and} \\ \partial^2_{xy} f = \partial^3_{xyz} f = 0 \\ \text{at the origin } (0,0,0). \end{array}}$$

NOTE:

WE REQUIRE MERELY

$(F_1, F_2, F_3) \in C^0$,

BUT OUR CONDITIONS

ON f INVOLVE DERIVATIVES

up to 3rd ORDER.

PASSING FROM HOCHSTER'S EXAMPLE TO THE GENERAL CASE, WE HAVE THE FOLLOWING RESULT.

Thm A:

Let $A_1, \dots, A_D$ be polys. on $\mathbb{R}^n$.

Fix $m \geq 0$.

Then there exist

LINEAR PARTIAL DIFF. OPERATORS

$L_1, \dots, L_K$

$\left\{ \text{WE CAN COMPUTE THEM IN PRINCIPLE} \right\}$

WITH THE FOLLOWING

PROPERTIES …

Each L_k has the form

$$L_k f(x) = \sum_{|\alpha| \leq \overline{m}} a_{k\alpha}(x)\, \partial^\alpha f(x)$$

where the coefficients $a_{k\alpha}$ are SEMIALGEBRAIC FNS.,

and maybe $\overline{m} > m$.

Let $f \in C^\infty(\mathbb{R}^n)$.

Then the EQUATION

$$A_1 F_1 + \cdots + A_D F_D = f$$

admits a C^m SOLUTION $(F_1, \dots, F_D)$

$$\Longleftrightarrow$$

$$L_1 f = L_2 f = \cdots = L_K f = 0 \text{ on } \mathbb{R}^n.$$

=

For instance, for Hochster's eq.

$$x^2 F_1 + y^2 F_2 + xyz^2 F_3 = f,$$

the $L_1, \dots, L_K$ are the differential operators

$$\boxed{\begin{gathered} \mathbb{1}_{x=y=0}\,\partial^0, \quad \mathbb{1}_{x=y=0}\,\partial_x, \quad \mathbb{1}_{x=y=0}\,\partial_y, \\ \mathbb{1}_{x=y=z=0}\,\partial^2_{xy}, \quad \mathbb{1}_{x=y=z=0}\,\partial^3_{xyz} \end{gathered}}$$

($\mathbb{1}_{\dots}$ DENOTES INDICATOR FN.

∂^0 DENOTES "NO DIFFERENTIATION")

WE SAY JUST A FEW WORDS ABOUT THE PROOF OF THM A, FOLLOWED BY A FEW WORDS ABOUT ITS APPLICATION TO Q2

(FINDING GENERATORS for poly. ideal)

To prove Thm A,

we study bundles

depending on a

C^∞ function f.

Our bundles have the form

$$\boxed{\mathcal{H}_f = \left(T(x) J_x^{\bar{m}} f + I(x) \right)_{x \in \mathbb{R}^n},}$$

where:

$$J_x^{\bar{m}} f = \left[\begin{array}{l} \text{Taylor poly of degree } \bar{m} \\ \text{of } f \text{ at } x \end{array} \right]$$

$$f \in C^\infty(\mathbb{R}^n)$$

and ...

$$T(x): \left[\begin{array}{l}\text{POLYS OF} \\ \text{DEGREE} \\ \text{at most } \bar{m} \\ \text{on } \mathbb{R}^n\end{array}\right] \to \mathcal{P}(\mathbb{R}^n, \mathbb{R}^D)$$

MAYBE $\bar{m} > m$!

is a linear map,

depending SEMIALGEBRAICALLY

on $x \in \mathbb{R}^n$;

and also...

$I(x) \subset \mathcal{R}_x^D$ is an

$\mathcal{R}_x$ - SUBMODULE,

depending SEMIALGEBRAICALLY

on $x \in \mathbb{R}^n$.

The linear map $T(x)$ and the $\mathcal{O}_x$-submodule $I(x)$ are assumed to satisfy a condition (OMITTED HERE) called "CONSISTENCY WITH JET MULTIPLICATION".

For such bundles

$$\mathcal{H}_f = \left(T(x) J_x^{\bar{m}} f + I(x) \right)_{x \in E},$$

we characterize those $f \in C^\infty(\mathbb{R}^n)$ for which the Glaeser refinement of $\mathcal{H}_f$ has all its fibers NON-EMPTY.

Moreover, we show that if the Glaeser refinement of $\mathcal{H}_f$ has non-empty fibers, then that Glaeser refinement $\widetilde{(\mathcal{H}_f)}$ has the form ...

$$\left(\widetilde{\mathcal{H}_f}\right) = \left(\widetilde{T}(x)\, J_x^{\widetilde{\bar{m}}} f + \widetilde{I}(x)\right)_{x \in E}$$

with $\widetilde{T}(x)$, $\widetilde{I}(x)$

Satisfying all the conditions

we assumed for $T(x)$, $I(x)$.

MAYBE $\widetilde{\bar{m}} > \bar{m}$!

ITERATING THESE RESULTS

l_* TIMES, WE

CHARACTERIZE THOSE $f \in C^\infty$

for which the

STABLE GLAESER REFINEMENT

OF $\mathcal{H}_f$

HAS ALL ITS FIBERS NON-EMPTY.

Theorem A then

follows easily.

APPLYING THM. A

GIVEN POLYS. $A_1, \dots, A_D$.

WE WANT TO UNDERSTAND THE POLYNOMIALS f such that the equation

$$A_1 F_1 + \dots + A_D F_D = f$$

has a C^m solution $(F_1, \dots, F_D)$.

Thm A tells us that those f are precisely the polys that satisfy

$$L_1 f = \cdots = L_K f = 0 \text{ on } \mathbb{R}^n,$$

for a list of linear partial differential operators $L_1, \dots, L_K$ with SEMIALG. coefficients.

Given any list $L_1, \cdots, L_K$

of such linear partial diff. ops.

$\left\{ \begin{array}{c} \text{NOT NECESSARILY THOSE} \\ \text{ARISING FROM THM A} \end{array} \right\}$

we define $\mathcal{M}(L_1, \cdots, L_K)$

to be the ideal of all

polys f such that

$L_1(Qf) = \cdots = L_K(Qf) = 0$ on $\mathbb{R}^n$

for all polys Q.

Thm A exhibits

(in principle) a list of

partial diff. ops. $L_1, \dots, L_K$

such that:

$$\boxed{\begin{array}{l}\text{A given poly } f \text{ admits a } C^m \text{ solution} \\ \text{of } A_1 F_1 + \dots + A_D F_D = f \\ \iff f \in \mathcal{M}(L_1, \dots, L_K)\end{array}}$$

CONSEQUENTLY,

Q2 [FIND GENERATORS FOR THE IDEAL OF ALL POLYS f FOR WHICH $A_1 F_1 + \cdots + A_D F_D = f$ ADMITS A C^m SOLUTION]

REDUCES TO THE FOLLOWING PROBLEM OF COMPUTATIONAL ALGEBRA:

PROBLEM:

GIVEN LINEAR PARTIAL DIFF. OPS.

$L_1, \dots, L_K$,

COMPUTE GENERATORS

FOR THE IDEAL

$\mathcal{M}(L_1, \dots, L_K)$.

WE SOLVE THIS PROBLEM

OF COMPUTATIONAL ALGEBRA,

THUS ANSWERING Q2.

("WE" = cf & KEVIN LULI.

Today's results on Q1 & Q2

are joint work with Kevin.)

AN AMUSING SPECIAL CASE

CONSIDER THE SINGLE DIFF. OP.

$L = \mathbb{1}_E \partial^0$ ($E \subset \mathbb{R}^n$ SEMIALG.)

THEN $\mathcal{M}(L)$ IS THE IDEAL OF ALL POLYS THAT VANISH ON E.

SO, AS A SPECIAL CASE OF OUR WORK ON COMPUTATIONAL ALGEBRA, WE COMPUTE GENERATORS FOR THE IDEAL OF POLYS VANISHING ON A GIVEN SEMIALGEBRAIC SET.

KEVIN & I WERE SURPRISED

TO DISCOVER THAT

THERE WAS NO PREVIOUSLY

KNOWN ALGORITHM TO

DO THAT.

SIMULTANEOUSLY WITH OUR WORK,

$\boxed{\text{M. SAFEY EL DIN, Z.-H. YANG, \& L. ZHI}}$

FOUND A MORE EFFICIENT ALGORITHM TO COMPUTE GENERATORS FOR THE IDEAL OF POLYS VANISHING ON A GIVEN SEMIALGEBRAIC SET.

So MUCH FOR OUR QUESTIONS

Q1 & Q2

REGARDING Q3 :

RECALL, Q3 ASKS WHETHER AN EQUATION

$$A_1 F_1 + \cdots + A_D F_D = f$$

(with $A_1, \cdots, A_D$, f polys) ADMITS A C^m SEMIALGEBRAIC SOLUTION, PROVIDED IT ADMITS A C^m SOLUTION.

WE DON'T KNOW!

AN EXAMPLE DUE TO

KOLLÁR & NOWAK

shows that it may be

impossible to take

$F_1, \dots, F_D$ to be

C^m RATIONAL FUNCTIONS.

There's a WELL-KNOWN

OPEN PROBLEM on

SEMIALGEBRAIC SETS & FNS.

that sounds similar ....

Let $\overline{E} \subset \mathbb{R}^n$ and $E \subset \overline{E} \times \mathbb{R}^m$

be SEMIALGEBRAIC.

Suppose there exists a continuous

$f : \overline{E} \to \mathbb{R}^m$ such that

$(x, f(x)) \in E$ for all $x \in \overline{E}$.

Can we take such an f to be

CONTINUOUS & SEMIALGEBRAIC ?

(I HAVEN'T THE SLIGHTEST IDEA.)

LET'S LEAVE THE

SEMIALGEBRAIC

WORLD,

&

RETURN TO ANALYSIS.

ON TO THE NEXT TOPIC!

SELECTION PROBLEMS I

Fix $m, n, D \geq 1$.

Suppose we are given a finite set $E \subset \mathbb{R}^n$,

and suppose that for each $x \in E$ we're given a convex "TARGET" $K(x) \subset \mathbb{R}^D$.

A C^m SELECTION

of the family of targets

$$\mathcal{K} = (K(x))_{x \in E}$$

is a C^m map

$$F : \mathbb{R}^n \to \mathbb{R}^D$$

such that

$$F(x) \in K(x) \text{ for all } x \in E.$$

If $D=1$ and each target

$K(x)$ is a single point

$$\boxed{K(x) = \{f(x)\}}$$

then a C^m selection

of $(K(x))_{x \in E}$ is simply

a C^m function F s.t.

$F = f$ on E.

PROBLEM:

Given a family of convex targets

$\mathcal{K} = (K(x))_{x \in E}$ as above,

find a C^m SELECTION F,

whose norm $\|F\|_{C^m(\mathbb{R}^n, \mathbb{R}^D)}$

is as small as possible

up to a factor C.

DETERMINED BY m, n, D

FINITENESS THM:

Given $m, n, D \geq 1$,

there exist

$k = k(m, n, D)$

and

$C = C(m, n, D)$

for which the following holds:

Let $E \subset \mathbb{R}^n$ (finite),

and

for each $x \in E$,

let $K(x) \subset \mathbb{R}^D$

be a given convex set.

Suppose that for each $S \subset E$ with at most k points, there exists a C^m selection of $(K(x))_{x \in S}$ with norm ≤ 1.

Then there exists a

C^m selection of

$(K(x))_{x \in E}$

with norm $\leq C$.

$=$

The proof of this result yields also a FINITENESS THM. for

INTERPOLATION WITH CONSTRAINTS.

For instance, ...

Thm: Given $m, n \geq 1$,

there exist $k = k(m,n)$ & $C = C(m,n)$ for which the following holds:

Let $f : E \to \mathbb{R}$ with $E \subset \mathbb{R}^n$ FINITE.

Suppose that given any $S \subset E$ with at most k points, there exists $F^S \geq 0$ such that $F^S = f$ on S and $\|F^S\|_{C^m} \leq 1$.

Then there exists a C^m fn.

$F \geq 0$ such that

$F = f$ on E and $\|F\|_{C^m} \leq C$.

$=$

To prove these results, we adapt ideas from the $\boxed{\text{CLASSICAL FINITENESS THM FOR } C^m(\mathbb{R}^n).}$

Recall that we worked with σ's and Γ's to prove that Thm.

To study Sobolev Spaces, we abandoned the Γ's, and worked with the σ's.

$$\boxed{\text{Today we will abandon the } \sigma\text{'s, and work with } \Gamma\text{'s.}}$$

Here are the BASIC NOTIONS:

WE WORK WITH

$$\boxed{\mathcal{P} = \left[\begin{array}{l}\text{VECTOR SPACE of POLYS} \\ \text{of DEGREE} \le m-1 \text{ on } \mathbb{R}^n\end{array}\right]},$$

and set

$$\boxed{J_x(F) = \left[\begin{array}{l}(m-1)^{\underline{RST}} \text{ DEGREE TAYLOR POLY} \\ \text{OF } F \text{ at } x\end{array}\right]},$$

$$\boxed{P \odot_x Q = J_x(PQ)},$$

$$\boxed{\mathcal{R}_x = (\mathcal{P}, \odot_x) \quad \text{"RING OF JETS AT } x\text{"}}$$

A **SHAPE FIELD** is a family

of (possibly empty) convex sets

$\Gamma(x, M) \subset \mathcal{P}$,

indexed by points $x \in E$

and positive real numbers M,

such that

$$\boxed{M \leq M' \Rightarrow \Gamma(x, M) \subset \Gamma(x, M').}$$

A shape field

$$\vec{\Gamma} = \left(\Gamma(x, M) \right)_{\substack{x \in E \\ M > 0}}$$

is called

WHITNEY ℓ-CONVEX

WITH

WHITNEY CONSTANT C_w

if the following holds :

LET $M > 0$, $x_0 \in E$,

$P_1, P_2 \in \Gamma(x_0, M)$,

$Q_1, Q_2 \in \mathcal{P}$, $0 < \delta \leq 1$.

Suppose that

$$|\partial^\alpha (P_1 - P_2)(x_0)| \leq M \delta^{m-|\alpha|} \quad \&$$

$$|\partial^\alpha Q_i (x_0)| \leq \delta^{-|\alpha|} \quad (i = 1, 2)$$

for $|\alpha| \leq m-1$.

Suppose also that

$$Q_1 \odot_x Q_1 + Q_2 \odot Q_2 = 1 \text{ in } \mathcal{R}_x.$$

Then

$$Q_1 \odot_x Q_1 \odot_x P_1 + Q_2 \odot_x Q_2 \odot_x P_2$$

belongs to

$\Gamma(x_0, C_W M)$.

=

As in the proof of the "classical" finiteness thm for C^m,

WHITNEY t-CONVEXITY

will allow us to patch together solutions of LOCAL SELECTION PROBS. into a global solution, using

WHITNEY PARTITIONS of UNITY.

NOTE THAT WHITNEY f-CONVEXITY IS NOW A PROPERTY of the Γ's, WHEREAS IN OUR EARLIER DISCUSSIONS, IT WAS A PROPERTY of the σ's.

[IN THE PRESENT CONTEXT, THERE ARE NO σ's.]

NEXT, WE ADAPT THE NOTION OF AN $\mathcal{A}$-BASIS

Recall our earlier definition:

An $(\mathcal{A}, \delta, C_B)$-basis of a symmetric convex set $\sigma \subset \mathcal{P}$ is a family of polys $(P_\alpha)_{\alpha \in \mathcal{A}}$ [EACH $P_\alpha \in \mathcal{P}$] such that ...

$$\boxed{\begin{array}{l} |\partial^\beta P_\alpha(x_0)| \le C_B\, \delta^{|\alpha|-|\beta|} \\ \text{for } \alpha \in \mathcal{A},\ |\beta| \le m-1 \end{array}}$$

$$\boxed{\begin{array}{l} \partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha} \quad [\text{Kronecker}] \\ \text{for } \beta, \alpha \in \mathcal{A} \end{array}}$$

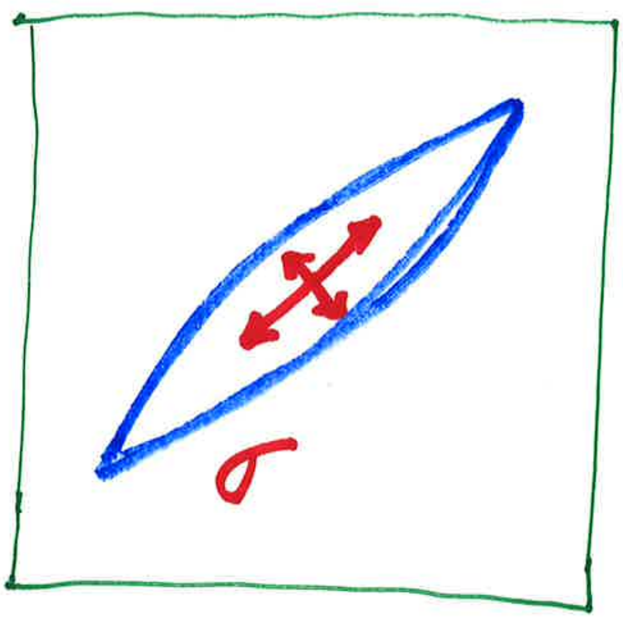
$$\boxed{\begin{array}{l} \delta^{m-|\alpha|} P_\alpha \in C_B\, \sigma \\ \text{for } \alpha \in \mathcal{A}. \end{array}}$$

Roughly speaking,

that means that σ is

THICK ENOUGH

in CERTAIN DIRECTIONS.



Today we have NO σ's,

only Γ's.

So we define an $\mathcal{A}$-basis

as follows.

Suppose that

$$\boxed{\begin{array}{l} \Gamma \subset \mathcal{P} \text{ convex} \\ x_0 \in \mathbb{R}^n \\ C_B, M_0, \delta > 0 \\ P_0 \in \Gamma \\ \mathcal{a} \subset \mathcal{M} \end{array}}$$

are given.

An $(\mathcal{A}, \delta, C_B)$ BASIS FOR Γ

at (x_0, M_0, P_0)

is a family of polynomials

$(P_\alpha)_{\alpha \in \mathcal{A}}$, each $P_\alpha \in \mathcal{P}$,

with the following properties:

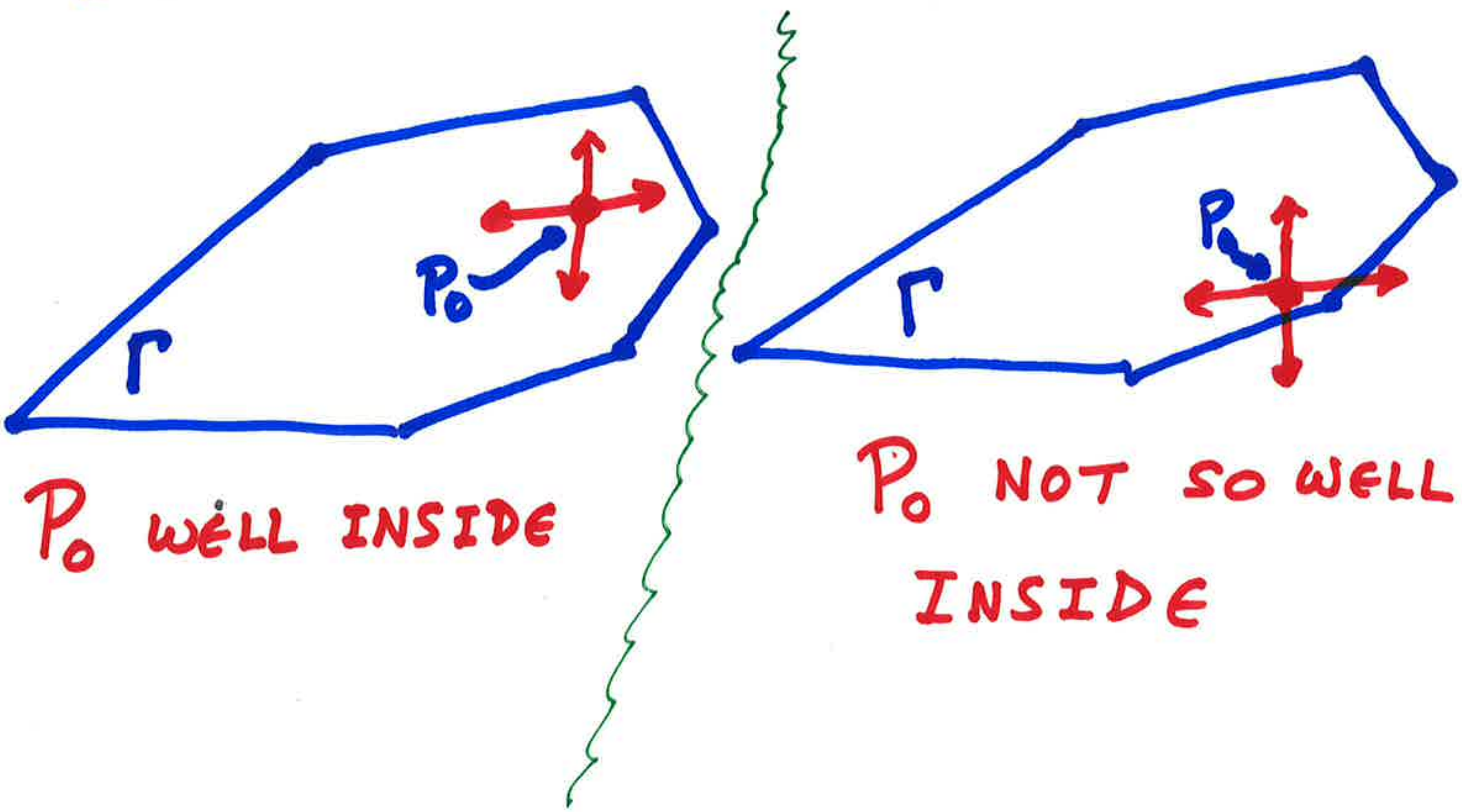
$$\boxed{\begin{array}{l} |\partial^\beta P_\alpha(x_0)| \le C_B\, \delta^{|\alpha|-|\beta|} \\ \text{for } \alpha \in \mathcal{A},\ |\beta| \le m-1. \end{array}}$$

$$\boxed{\begin{array}{l} \partial^\beta P_\alpha(x_0) = \delta_{\beta\alpha} \\ \text{for } \beta, \alpha \in \mathcal{A} \end{array}}$$

$$\boxed{\begin{array}{l} P_0 + \dfrac{M_0\,\delta^{m-|\alpha|}}{C_B} P_\alpha \text{ and } P_0 - \dfrac{M_0\,\delta^{m-|\alpha|}}{C_B} P_\alpha \\ \text{belong to } \Gamma \\ \text{for each } \alpha \in \mathcal{A}. \end{array}}$$

Roughly speaking, that means that Γ is thick enough in certain directions,

<u>and</u> that P_0 is well inside Γ.



P_0 WELL INSIDE

P_0 NOT SO WELL INSIDE

ARMED WITH THESE NOTIONS, WE CAN ADAPT THE PROOF OF THE CLASSICAL FINITENESS THM. FOR $C^m(\mathbb{R}^n)$, TO OBTAIN THE FOLLOWING RESULT FOR SHAPE FIELDS.

FINITENESS THM FOR SHAPE FIELDS:

Let $m, n \geq 1$.

For a large enough $k = k(m,n)$, the following holds:

Let $\vec{\Gamma} = (\Gamma(x, M))_{\substack{x \in E \\ M > 0}}$ be a Whitney t-convex shape field with Whitney constant C_W.

Let $M_0 > 0$ be given.

Suppose that for any $S \subset E$

with at most k points,

there exists $F^S \in C^m(\mathbb{R}^n)$

with norm $\leq M_0$,

such that

$J_x(F^S) \in \Gamma(x, M_0)$

for all $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n)$, with norm $\leq C M_0$, such that

$J_x(F) \in \Gamma(x, C M_0) \;\; \forall x \in E.$

Here, C depends only on m, n and the Whitney const. C_W.

Except for one EXCITING MOMENT,

the proof of the

FINITENESS THM for SHAPE FIELDS

is adapted from the proof of the

FINITENESS THM for $C^m(\mathbb{R}^n)$.

Whenever the classical proof mentions Whitney t-convexity or an $\mathcal{A}$-basis for σ, we now simply substitute the corresponding notion (explained above) for Γ.

There is one KEY POINT in the proof where the classical argument breaks down and we have to introduce a NEW IDEA with No ANALOGUE in the classical case.

The new idea is simple,

But we would have to

Delve deeply into the

details of the proof

To see what breaks down

and how to fix it.

Please just trust me!

ONCE WE KNOW THE

FINITENESS THM FOR SHAPE FIELDS

WE CAN EASILY DEDUCE THE

FINITENESS THMS. for

$\boxed{\text{$C^m$ SELECTION \& for NON-NEGATIVE C^m INTERPOLATION.}}$

NOTE THAT THE FINITENESS THM. FOR SHAPE FIELDS DEALS WITH SCALAR-VALUED FUNCTIONS, WHILE THE FINITENESS THM. FOR C^m SELECTION DEALS WITH $\mathbb{R}^D$- VALUED FUNCTIONS.

WE REDUCE PROBLEMS ON

VECTOR-VALUED FUNCTIONS

TO PROBLEMS ON

SCALAR-VALUED FUNCTIONS

by the SAME TRICK used

to treat BUNDLES with

respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$.

IN PASSING FROM

INTERPOLATION PROBLEMS

TO SHAPE FIELDS,

WE'VE GAINED

GENERALITY,

BUT WE'VE LOST

PLENTY ...

FOR SHAPE FIELDS,

$\boxed{\text{WE HAVE A FINITENESS THM, BUT NOT A REFINED FINITENESS THM.}}$

$\boxed{\text{WE DON'T HAVE A LINEAR EXTENSION OPERATOR}}$

MOST SERIOUSLY,

FOR SHAPE FIELDS,

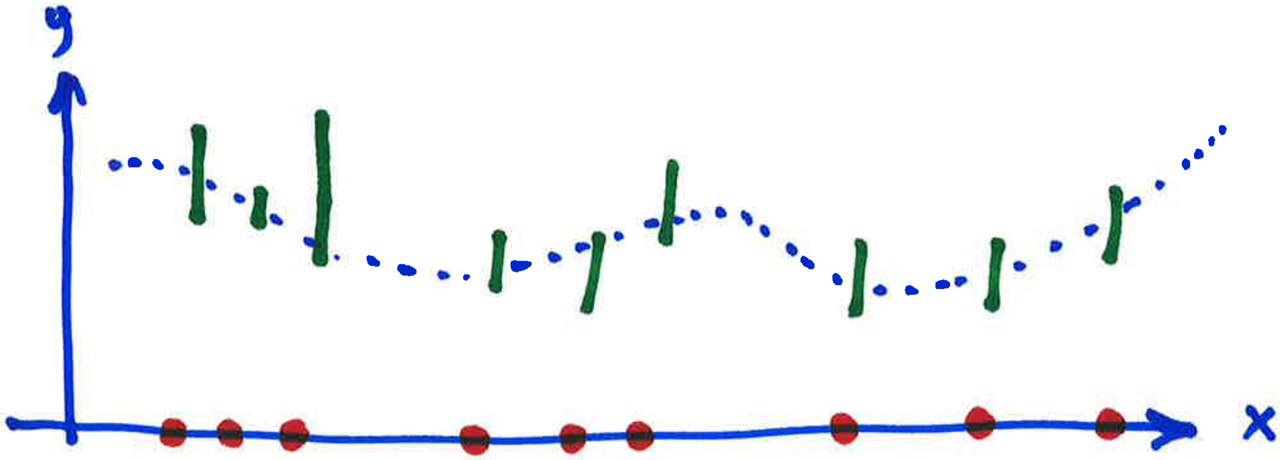
WE HAVE NO ANALOGUES

OF THE CLASSICAL

RESULTS ON

EFFICIENT COMPUTATION.

A SIMPLE EXAMPLE OF A SELECTION PROBLEM FOR C^2 MAPS $F : \mathbb{R}^1 \to \mathbb{R}^1$



E = SET OF RED DOTS

$\#(E) = N$

For each $x \in E$,

$K(x) \subset \mathbb{R}^1$

is the GREEN INTERVAL

LYING OVER x.

WE WANT TO COMPUTE A FN.

$F \in C^2(\mathbb{R}^1)$

whose GRAPH PASSES THROUGH

the "ERROR BARS",

WITH $\|F\|_{C^2(\mathbb{R}^1)}$

AS SMALL AS POSSIBLE,

UP TO A UNIVERSAL

CONSTANT FACTOR.

WE WANT TO FIND

$F(x), F'(x)$ (all $x \in E$)

FOR SUCH AN F,

USING $O(N \log N)$

Computer operations.

Can that be done?

WE DON'T KNOW,

EVEN for simple

ONE-DIMENSIONAL PROBLEMS

LIKE THE ONE PICTURED ABOVE.

REGARDING COMPUTATION,

HERE'S THE BEST WE CAN DO

SO FAR.

WE WILL MAKE OUR

SELECTION PROBLEM

EASIER

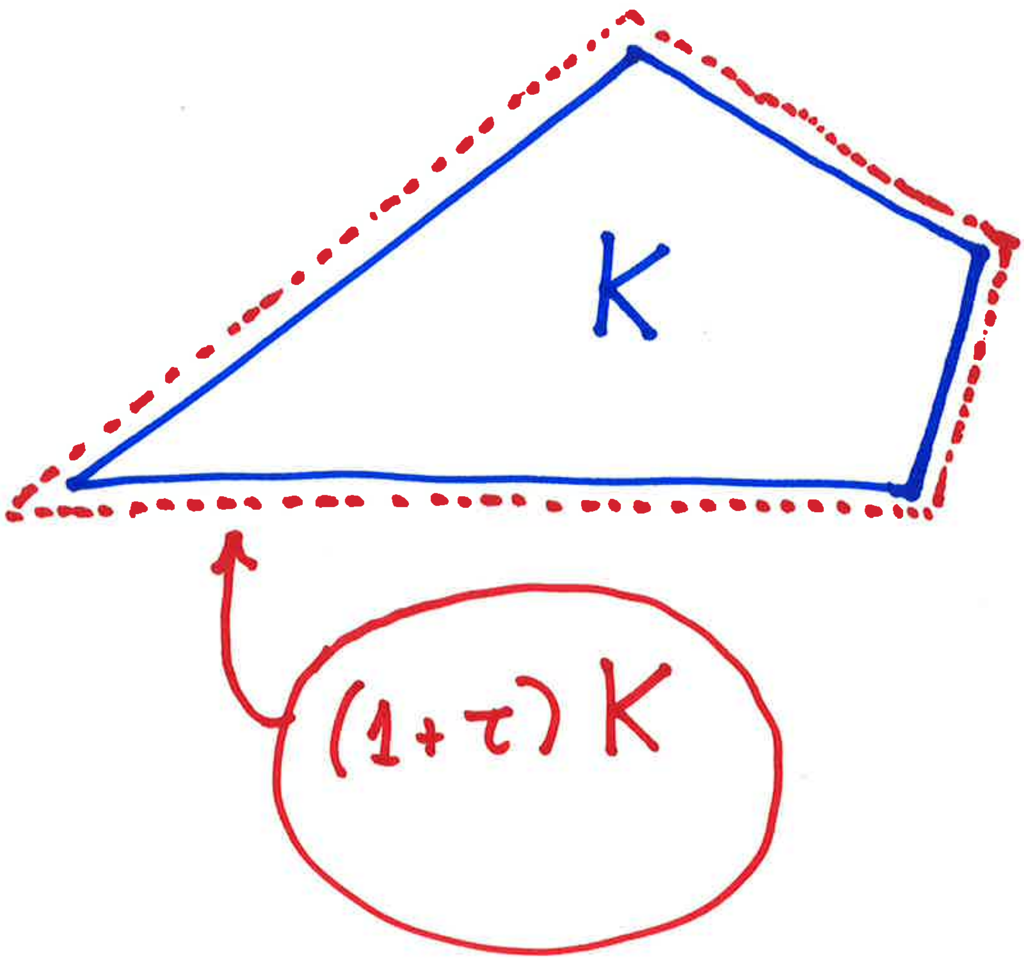
BY SLIGHTLY ENLARGING

THE TARGETS $K(x)$.

Let $K \subset \mathbb{R}^D$ be a compact, convex set.

Let $\tau > 0$.

Then $(1+\tau) \diamond K$ denotes the convex set obtained by dilating K about its center of mass by a factor $(1+\tau)$.


K
(1+τ) K

APPROXIMATE SELECTION ALGORITHM

Fix $m, n, D \geqslant 1$.

GIVEN :

- A FINITE SET $E \subset \mathbb{R}^n$ $(\#(E) = N)$

- For each $x \in E$, a compact convex polytope $K(x) \subset \mathbb{R}^D$, defined by $\leq S$ LINEAR CONSTRAINTS.

- Positive real numbers M, τ.

Given the above,

we produce one of the

following two OUTCOMES,

using at most $C(\tau, S)\, N \log N$

Computer operations.

[$C(\tau, S)$ depends only on τ, S, m, n, D]

THE OUTCOMES:

OUTCOME 1 ("NO GO"):

WE GUARANTEE THAT

THERE DOES NOT EXIST

$F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$

WITH NORM $\leq M$ such that

$F(x) \in K(x)$ for all $x \in E$.

OUTCOME 2 ("SUCCESS"):

WE PRODUCE A FUNCTION $f : E \to \mathbb{R}^D$ satisfying

$f(x) \in (1+\tau) \diamond K(x)$ for all $x \in E$,

and WE GUARANTEE that there exists $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ with norm $\leq CM$ s.t. $F = f$ on E.

[DEPENDS ONLY ON m, n, D]

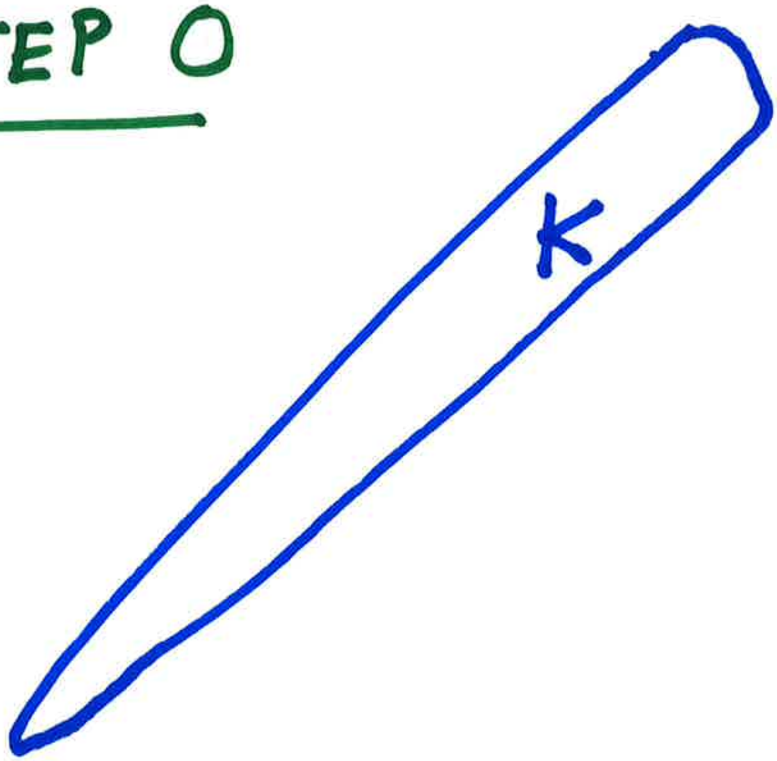
In the event of Success, our classical interpolation algorithm computes a function $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$ with norm $\leq CM$, such that $F = f$ on E and therefore

$F(x) \in (1+\tau) \diamond K(x) \quad \forall x \in E$.

The algorithm combines ideas from the classical interpolation algorithm with ideas from the proof of the FINITENESS THM for SHAPE FIELDS.

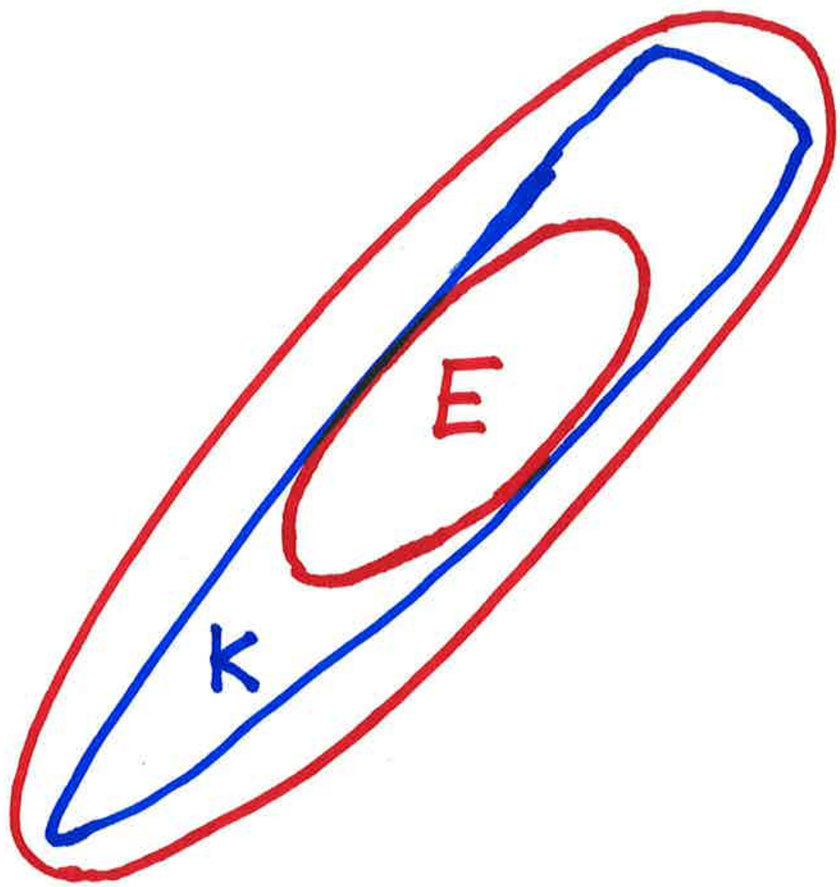
HOW WE REPRESENT & SIMPLIFY CONVEX SETS

STEP 0



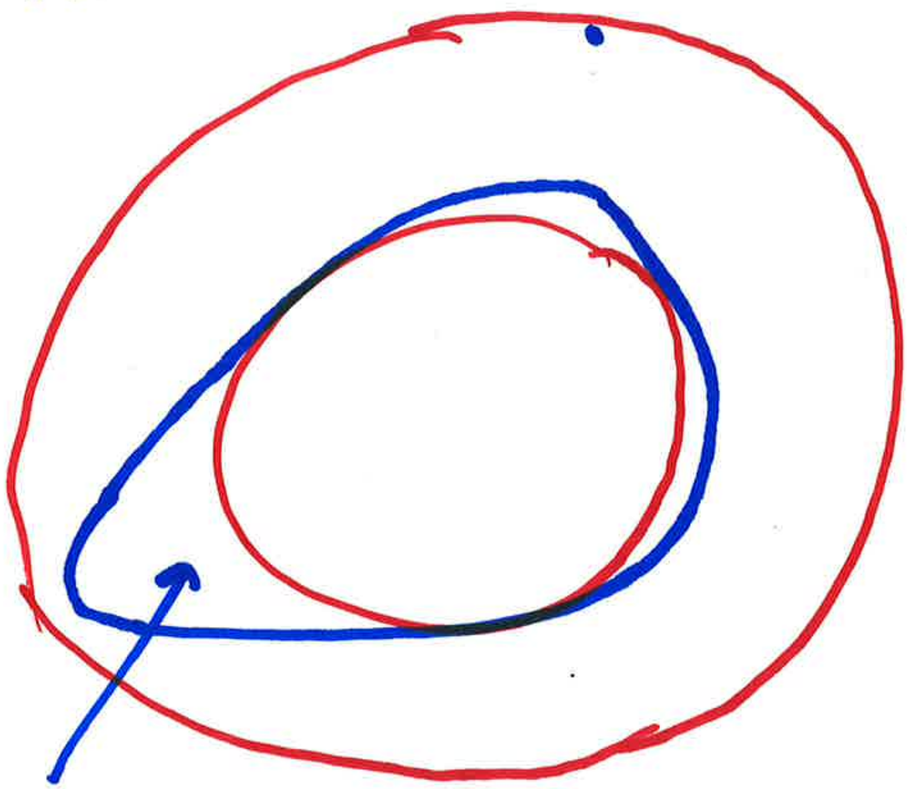
STEP 1 :

K and its JOHN ELLIPSOID



STEP 2:

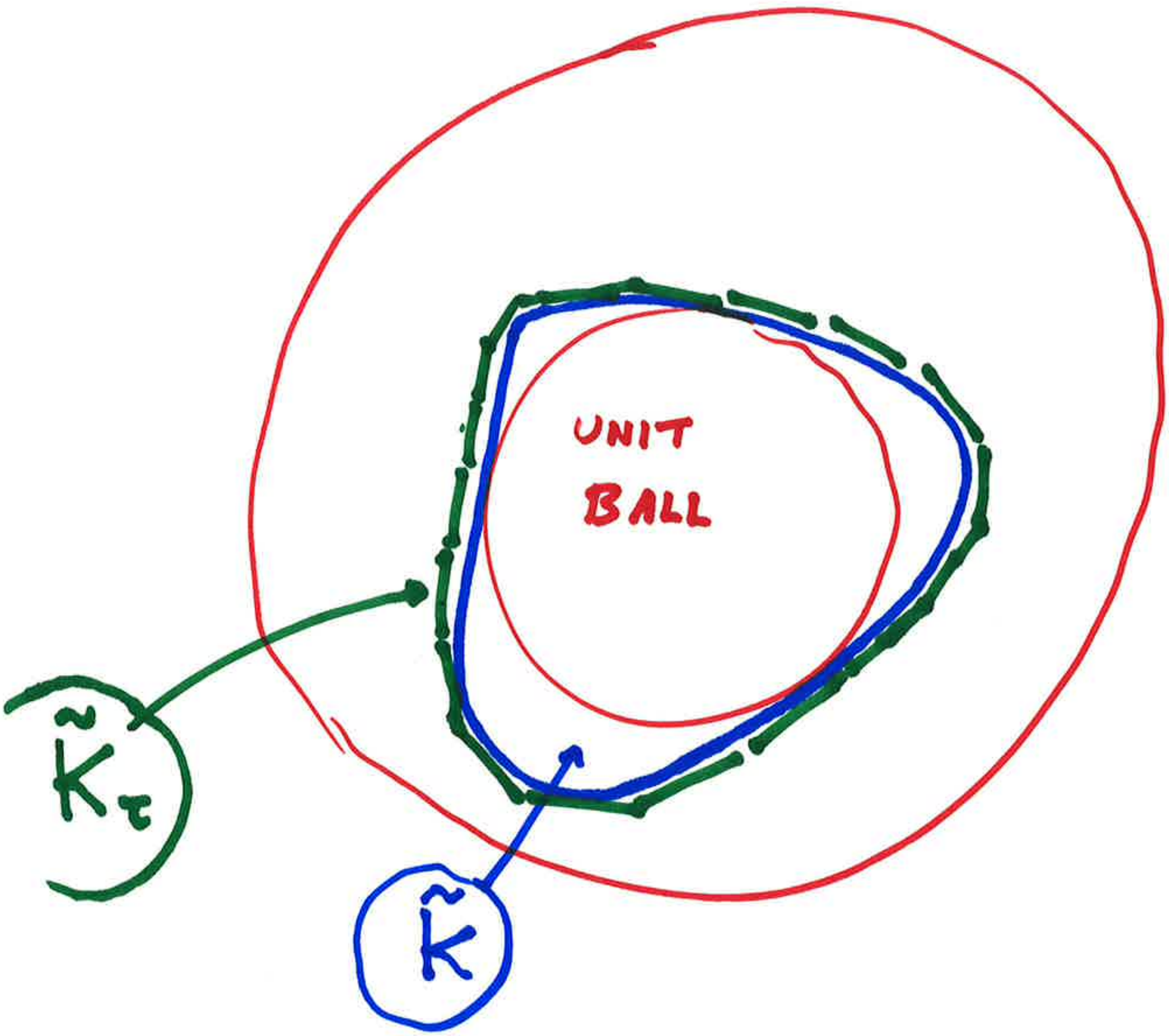
APPLY AN AFFINE MAP A TO BRING THE JOHN ELLIPSOID TO THE UNIT BALL



$\tilde{K}=AK$

STEP 3:

APPROXIMATE $\tilde{K} = AK$ BY A (τ-DEPENDENT) CONVEX POLYTOPE $\tilde{K}_\tau$, DEFINED BY AT MOST $C(\tau)$ LINEAR CONSTRAINTS.



$$\tilde{K} \subset \tilde{K}_\tau \subset (1+\tau) \diamond \tilde{K}$$

BEFORE LEAVING C^m SELECTION,

LET ME GIVE

PROPER CREDIT.

The Finiteness Thms

for Shape Fields,

C^m Selection, &

Non-Negative Interpolation

are due to

cf, Arie Israel, Kevin Luli

THE APPROXIMATE

C^m SELECTION

ALGORITHM is

due to cf and

BERNAT GUILLEN PEGUEROLES.

SELECTION PROBLEMS II:

LIPSCHITZ SELECTION.

So far, we've studied functions F defined on $\mathbb{R}^n$.

Now we study functions defined on an arbitrary metric space (X, d).

SETUP: Fix $D \geq 1$.

We are given a metric space (X, d).

For each $x \in X$ we are given a compact convex set $K(x) \subset \mathbb{R}^D$.

A SELECTION OF THE FAMILY

$\mathcal{K} = (K(x))_{x \in X}$

is a map $F : X \to \mathbb{R}^D$

such that

$F(x) \in K(x)$ for all $x \in X$.

If that F is Lipschitz then

we call it a LIPSCHITZ SELECTION.

FINITENESS THM FOR LIPSCHITZ SELECTION:

Given (X, d), D, $(K(x))_{x \in X}$

as above.

Suppose that for any $S \subset X$ with $\#(S) \leq 2^D$, there exists $F^S : S \to \mathbb{R}^D$ with Lipschitz const ≤ 1 such that

$F^S(x) \in K(x)$ for all $x \in S$.

Then there exists

$F : X \to \mathbb{R}^D$,

with Lipschitz const. $\leq C(D)$,

such that

$F(x) \in K(x)$ for all $x \in X$.

Here, $C(D)$ depends only

on D.

The result holds more generally.

X can be a "pseudometric space", i.e., $d(x, y)$ can be 0 or ∞.

Instead of taking $K(x) \subset \mathbb{R}^D$, we can fix a Banach space Y and assume that each $K(x)$ is contained in a D-dimensional affine subspace of Y.
(Then replace 2^D by 2^{D+1}.)

The numbers 2^D, 2^{D+1} in the two flavors of the Finiteness Thm for Lip. Selection are **Best Possible**.

If the metric space (X, d) is <u>finite</u>, then we needn't assume that the convex sets $K(x)$ are compact.

A VARIANT OF A SPECIAL CASE OF THE FINITENESS THM FOR LIPSCHITZ SELECTION WAS THE CRUCIAL INGREDIENT IN THE ORIGINAL WORK OF BRUDNYI - SHVARTSMAN, & SHVARTSMAN ON

$C^2(\mathbb{R}^n)$, $C^{1,\omega}(\mathbb{R}^n)$, ...

Today, we use ideas

from the study of $C^m(\mathbb{R}^n)$

to attack

Lipschitz Selection Problems.

Today's

FINITENESS THM FOR

LIPSCHITZ SELECTION

is due to

cf & PAVEL SHVARTSMAN.

SKETCH OF PROOF OF THE FINITENESS THM FOR LIPSCHITZ SELECTION

There are 4 Main Steps.

STEP 1:

PROVE THE RESULT FOR THE SPECIAL CASE IN WHICH (X, d) IS A METRIC TREE.

DON'T WORRY ABOUT THE SHARP "FINITENESS CONST." 2^D — JUST GET A RESULT FOR A FINITENESS CONST. DEPENDING ONLY ON D.

STEP 2:

USE OUR RESULT ON METRIC TREES TO STUDY GENERAL METRIC SPACES.

For each $x \in X$, DEFINE A CORE $K^*(x) \subset K(x)$, S.T.

$$\boxed{\text{HAUSDORFF DISTANCE}\left(K^*(x), K^*(y)\right) \leq C(D) \cdot d(x, y)}$$

for $x, y \in X$.

STEP 3:

DEFINE A SELECTION $F: X \to \mathbb{R}^D$ by SETTING

$F(x) =$ STEINER POINT of $K^*(x)$.

This proves the FINITENESS THM for LIP. SELECTION, except that the sharp FINITENESS CONST. 2^D is replaced by a large const. k depending only on D.

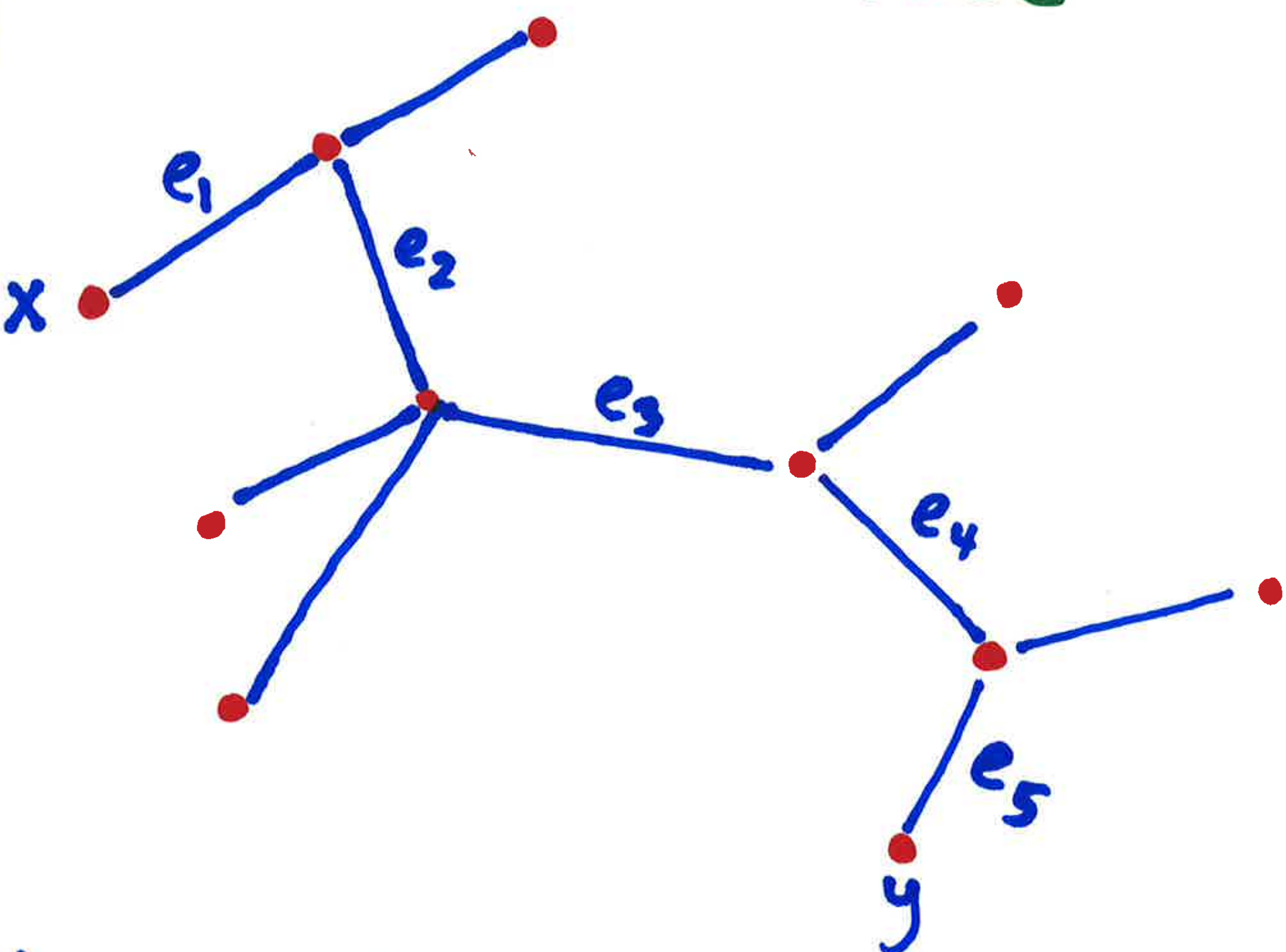
STEP 4 :

Obtain the sharp finiteness constant 2^D, completing the proof.

LET US BRIEFLY EXPLAIN
EACH OF THOSE 4 STEPS.

STEP 1 : PROVING THE FINITENESS THM. for METRIC TREES.

A METRIC TREE



$$d(x,y) = \Delta(e_1) + \Delta(e_2) + \Delta(e_3) + \Delta(e_4) + \Delta(e_5)$$

SURPRISE:

METRIC TREES CAN BE TREATED LIKE $\mathbb{R}^n$.

CRUCIAL PROPERTY of $\mathbb{R}^n$:

Given $\delta > 0$, we can partition $\mathbb{R}^n$ into cubes Q_ν of diameter δ, so that at most C of the Q_ν intersect any given ball of radius $c\delta$.

That CRUCIAL PROPERTY of $\mathbb{R}^n$ lets us define Calderón-Zygmund decompositions, Whitney partitions of unity, ...

A SIMILAR PROPERTY HOLDS FOR METRIC TREES.

(They have "NAGATA DIMENSION" = 1.)

HERE'S THE STATEMENT:

LET (X, d) be a METRIC TREE.

$$\boxed{\text{Given } s>0, \text{ we can partition } X \text{ into sets } Q_\nu \text{ of diameter } \le s, \text{ such that any ball of radius } \frac{s}{16} \text{ meets at most } 2 \text{ of the } Q_\nu.}$$

We have

$\boxed{\text{No TIME to show the simple construction of the } Q_\nu}$

and

$\boxed{\text{No TIME to discuss how to adapt our proof of the FINITENESS THM from } \mathbb{R}^n \text{ to a metric tree.}}$

LET'S JUST DECLARE VICTORY OVER STEP 1.

STEP 2: DEFINING THE CORE

Given a metric space (X, d).

Given $K(x) \subset \mathbb{R}^D$ compact convex for each $x \in E$.

Assume, for large enough $k = k(D)$, that

$\boxed{\begin{array}{l}\text{For any } S \subset X \text{ with } \#(S) \le k \\ \text{there exists } F^S : S \to \mathbb{R}^D \text{ s.t.} \\ F^S(x) \in K(x) \text{ for all } x \in S \\ \text{and} \\ |F^S(x) - F^S(y)| \le d(x,y) \\ \text{for all } x, y \in S.\end{array}}$

For each $x \in X$ we will define a

CORE $K^*(x) \subset K(x)$

such that

$$\boxed{\begin{array}{l}\text{HAUSDORFF DIST.}\left(K^*(x), K^*(y)\right) \\ \qquad \leq C(D) \cdot d(x,y) \\ \text{for } x, y \in X.\end{array}}$$

To do so, we use our result on METRIC TREES (STEP 1).

Let T be any tree.

(We don't yet have a metric on T.)

A map $\varphi : T \to X$ will be called

ADMISSIBLE

if $\varphi(t_1) \neq \varphi(t_2)$

whenever t_1 & t_2 are

joined by an edge.

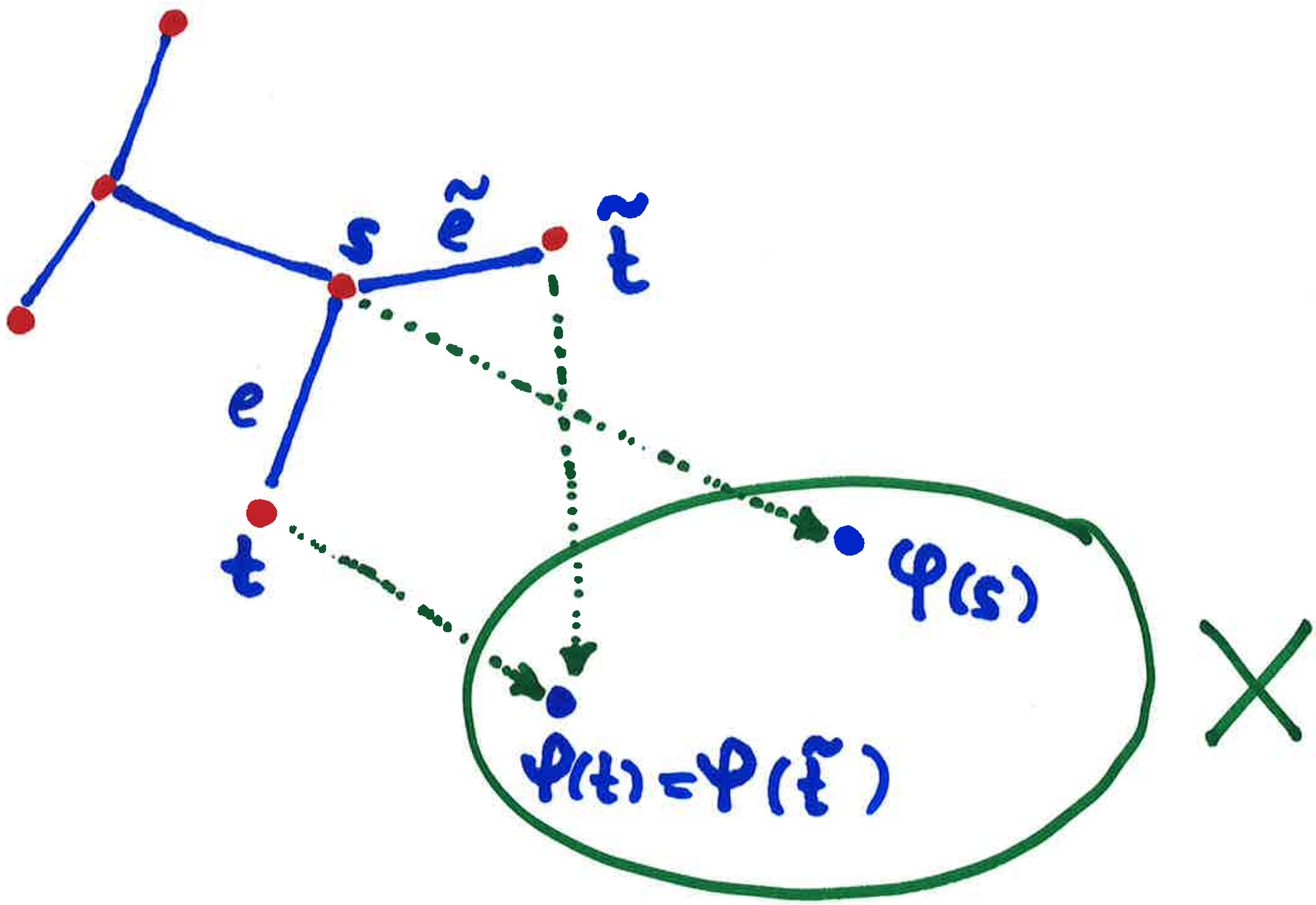
If φ is admissible, then we can assign a length to each edge of T by setting

$$\Delta(e) = d(\varphi(t_1), \varphi(t_2)),$$

where t_1 & t_2 are the two nodes joined by e.

These $\Delta(e)$ define a tree metric on T. Let's call that metric d_φ.

AN EXAMPLE OF AN ADMISSIBLE MAP AND THE TREE METRIC IT DEFINES



$$\Delta(e) = d(\varphi(t), \varphi(s))$$

$$\Delta(\tilde{e}) = d(\varphi(\tilde{t}), \varphi(s))$$

$$d_\varphi(t, \tilde{t}) = \Delta(e) + \Delta(\tilde{e})$$

WE CAN PULL BACK OUR

LIP. SELECTION PROBLEM

$(X, d) \quad (K(x))_{x \in X}$

TO A LIP. SELECTION PROBLEM

for the TREE METRIC

(T, d_φ)

by SETTING

$K_\varphi(t) = K(\varphi(t))$ for $t \in T$.

By the result of STEP 1

(the FINITENESS THM for METRIC TREES)

there exists a map

$F : T \to \mathbb{R}^D$ such that

$F(t) \in K_\varphi(t) = K(\varphi(t))$ for all $t \in T$

and

$|F(s) - F(t)| \leq C(D) \cdot d_\varphi(s,t)$

for all $s, t \in T$.

Let's call such an F

an

ADMISSIBLE SELECTION

for (T, φ).

$$\boxed{\text{So ADMISSIBLE SELECTIONS exist whenever } T \text{ is a tree and } \varphi: T \to X \text{ is an ADMISSIBLE MAP.}}$$

Now suppose we pick BASE POINTS $t_0 \in T$ and $x_0 \in X$ such that $\varphi(t_0) = x_0$.

Then we define the "ORBIT"

$$\boxed{\mathcal{O}(T, \varphi, t_0, x_0) = \{ F(t_0) : F \text{ any admissible selection for } (T, \varphi) \}}$$

Thanks to STEP 1, WE KNOW that the orbit $O(T, \varphi, t_0, x_0)$ is NON-EMPTY.

The **Core** $K^*(x_0)$ is defined to be the intersection of $\mathcal{O}(T, \varphi, t_0, x_0)$ over all

$$\boxed{\begin{array}{l}\text{TREES } T, \\ \text{ADMISSIBLE MAPS } \varphi : T \to X \\ \text{AND POINTS } t_0 \in \varphi^{-1}(x_0)\end{array}}$$

BECAUSE ORBITS ARE NON-EMPTY, ONE CAN SHOW EASILY

that

$$\boxed{K^*(x) \subset K(x) \text{ IS A NON-EMPTY COMPACT CONVEX SET } (\text{all } x \in X)}$$

and that

$$\boxed{\text{HAUSDORFF DIST}\left(K^*(x), K^*(y)\right) \leq C(D) \cdot d(x,y) \quad \text{for } x, y \in X}$$

So we have defined the CORE.

LET'S DECLARE VICTORY

over STEP 2.

STEP 3: The STEINER POINT

Let $K \subset \mathbb{R}^D$ be compact, convex.

$$\boxed{\text{For } R>0, \text{ let } K_R \text{ denote the set of points within distance } R \text{ from } K.}$$

Let v_R denote the center of mass of K_R.

As $R \to \infty$,

v_R tends to a limit v_∞,

and v_∞ belongs to K.

v_∞ is called the STEINER POINT of K.

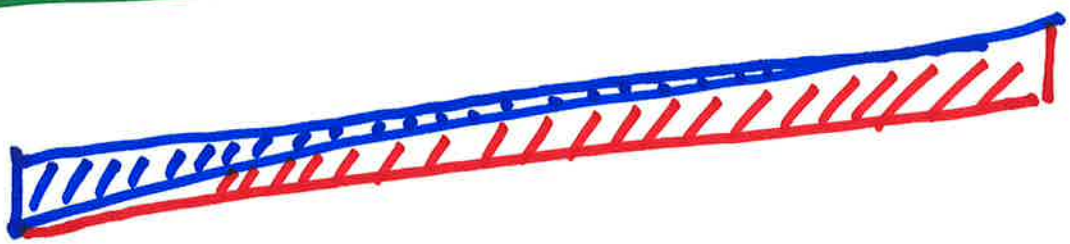
Standard Fact from Convex Geometry:

Let $K, K' \subset \mathbb{R}^D$ be compact convex sets, with Steiner pts. x, x', respectively.

Then $|x - x'| \leq C(D) \cdot d_H(K, K')$

where d_H denotes Hausdorff Dist. and $C(D)$ depends only on D.

NOTE: Can't use e.g. the center of mass in place of the Steiner pt.



Here are K, K' with small Hausdorff dist., but with their centers of mass far apart.

APPLICATION TO LIP. SELECTION

Given a metric space (X, d),

and given a compact convex $K(x) \subset \mathbb{R}^D$

for each $x \in X$,

we have defined a CORE $K^*(x) \subset K(x)$

for each $x \in X$.

We know that each $K(x)$ is a non-empty compact convex set, and we know that

$$d_H\left(K(x), K(y)\right) \leq C(D) \cdot d(x,y)$$

for $x, y \in X$.

WE NOW DEFINE

$F(x) =$ STEINER PT. OF $K^*(x)$

for each $x \in X$.

For $x \in X$, we then have

$$\boxed{F(x) \in K^*(x) \subset K(x).}$$

For $x, y \in X$, we have

$$\boxed{\begin{aligned} |F(x) - F(y)| &\leq C(D)\, d_H(K^*(x), K^*(y)) \\ &\leq C'(D) \cdot d(x,y). \end{aligned}}$$

So F is our desired Lipschitz Selection.

VICTORY OVER STEP 3 !

P.S. In the variant of the Finiteness Thm in which $\mathbb{R}^D$ is replaced by a Banach space Y, we need a "Steiner-Type Point" in place of the classical Steiner point.

Pavel Shvartsman proved the existence of such a point.

STEP 4: The Sharp Finiteness Const.

So far, we know the following,

for a large enough

$k^{\#} = k^{\#}(D)$ and $C^{\#} = C^{\#}(D)$:

Let $(K(x))_{x \in X}$ be a family of compact convex subsets of $\mathbb{R}^D$, parametrized by the points of a metric space (X, d).

Let $M > 0$.

Suppose that for each $S \subset X$

with $\#(S) \leq k^{\#}$

there exists $F^S : S \to \mathbb{R}^D$

with Lipschitz const. $\leq M$,

such that

$F^S(x) \in K(x)$ for all $x \in S$.

Then there exists

$F : X \to \mathbb{R}^D$

with Lipschitz const. $\leq C^{\#} M$,

such that

$F(x) \in K(x)$ for all $x \in X$.

$=$

We must show that

$$\boxed{k^{\#} \text{ can be taken to be } 2^{D}.}$$

To prove this, the KEY STEP

is as follows...

Thm: Let (Z, d_Z) be a FINITE metric space.

Let $(K(x))_{x \in Z}$ be a family of compact convex subsets of $\mathbb{R}^D$, parametrized by the points of Z.

Suppose that for any $S \subset Z$ with $\#(S) \leq 2^D$ there exists $F^S : S \to \mathbb{R}^D$ with Lip. constant ≤ 1, such that $F^S(x) \in K(x)$ for all $x \in S$.

Then there exists

$F: Z \to \mathbb{R}^D$

such that

- $F(x) \in K(x)$ for all $x \in Z$,

and

- The Lipschitz const. of F is at most $C(D, \#(Z))$.

THE PROOF OF THAT THM.

IS SIMPLE & CLEVER,

BUT

NO TIME TO EXPLAIN!

Let's BELIEVE IT.

THE COUP DE GRÂCE

Given $(K(x))_{x \in X}$ as usual.

Assume that for any $S \subset X$ with $\#(S) \leq 2^D$ there exists $F^S : S \to \mathbb{R}^D$ with Lipschitz const ≤ 1, s.t. $K(x) \ni F^S(x)$ for all $x \in S$.

Then, by the Thm just cited, the following holds.

$$\boxed{\begin{array}{l}\text{Let } k^\# \text{ be as in Step 3.}\\ \text{Let } Z \subset X \text{ with } \#(Z) \le k^\#.\\ \text{Then there exists } F^Z : Z \to \mathbb{R}^D,\\ \text{with Lipschitz const.} \le C(D, k^\#),\\ \text{such that}\\ F^Z(x) \in K(x) \text{ for all } x \in Z.\end{array}}$$

That's precisely the hypothesis of our result from STEP 3, with $M = C(D, k^{\#})$.

Applying our result from Step 3, we obtain a Lipschitz selection of $(K(x))_{x \in X}$, with Lipschitz const. at most $C^{\#} \cdot C(D, k^{\#})$.

However, $C^{\#}$ and $k^{\#}$ are determined by D.

So our bound $C^{\#} \cdot C(D, k^{\#})$ also depends only on D.

That proves the FINITENESS THM FOR LIP. SELECTION, with the sharp finiteness constant 2^D.

So MUCH FOR

LIPSCHITZ

SELECTION!

LET'S FINISH THE

LECTURE SERIES

BY STATING

A FEW

OPEN PROBLEMS

We've seen most of them already.

Many of the QUESTIONS are not due to me.

THE QUESTIONS

PROVE OR DISPROVE

EXISTENCE OF

LINEAR EXTENSION OP's

$$T : L^{m,p}(E) \to L^{m,p}(\mathbb{R}^n)$$

when $\frac{n}{m} < p \leq n$ (and $p > 1$).

FIND EFFICIENT ALGORITHMS

FOR C^m SELECTION

&

FOR INTERPOLATION

by NON-NEGATIVE FUNCTIONS

How can one decide whether a given fn. $\varphi : E \to \mathbb{R}$ ($E \subset \mathbb{R}^n$ COMPACT, MAYBE INFINITE) extends to a C^∞ function on $\mathbb{R}^n$?

REMARKABLE EXAMPLES DUE TO W. PAWŁUCKI SHOW THAT φ can extend to a C^m function for any given m, but NOT to a C^∞ function.

Let $\mathcal{H} = (H(x))_{x \in E}$ be a bundle with respect to $C^m(\mathbb{R}^n, \mathbb{R}^D)$.

Suppose $H(x)$ depends SEMIALGEBRAICALLY on x,

AND SUPPOSE $\mathcal{H}$ HAS A SECTION.

MUST $\mathcal{H}$ HAVE A SEMIALGEBRAIC SECTION?

A SPECIAL CASE

LET $\varphi: E \to \mathbb{R}$ BE SEMIALGEBRAIC, and let $m \geq 1$.

IF φ EXTENDS TO A C^m FUNCTION, THEN MUST IT EXTEND TO A SEMIALGEBRAIC C^m FUNCTION?

LET $\Omega \subset \mathbb{R}^n$ BE OPEN

(& CONNECTED).

HOW CAN WE TELL WHETHER EVERY $F: \Omega \to \mathbb{R}$ s.t.

$$\int_\Omega |\partial^\alpha F|^p \, dx < \infty \text{ for } |\alpha| \le m$$

Extends to a function in $W^{m,p}(\mathbb{R}^n)$?

GIVE INTERPOLATION ALGORITHMS THAT CAN BE USED IN PRACTICE,

i.e., LOWER THE IMPLICIT CONSTANTS.

THE RECENT

CO-ORDINATE - FREE

PROOF OF THE

FINITENESS THM for C^m

IS A PROMISING

FIRST STEP.

Given $f : E \to \mathbb{R}$, $E \subset \mathbb{R}^n$,

$\#(E) = N < \infty$, as usual.

Given $m \geq 1$.

Given also $\varepsilon > 0$.

Compute an interpolant F whose C^m norm is as small as possible, up to a factor $(1+\varepsilon)$.

Can we do so using

ONE-TIME WORK $\leq C(\varepsilon) N \log N$

&

QUERY WORK $\leq C(\varepsilon) \log N$?

This question requires that we specify precisely which norm we use on C^m,

e.g.

$$\sup_{x\in\mathbb{R}^n}\left(\sum_{|\alpha|\leq m}|\partial^\alpha F(x)|^2\right)^{1/2}$$

or

$$\sup_{x\in\mathbb{R}^n}\max_{|\alpha|\leq m}|\partial^\alpha F(x)|.$$

MAKE ANY REASONABLE CHOICE!

For this problem,

the natural analogue

of the

FINITENESS THM.

is

FALSE !

(cf & Bo'az Klartag —

the "ZIGZAG" counterexample)

NEVERTHELESS, THE DESIRED ALGORITHM EXISTS FOR $C^2(\mathbb{R}^2)$, THE SIMPLEST CASE IN WHICH THE FINITENESS THM FAILS.

UNDERSTAND INTERPOLATION

by

CONVEX FNS. [AZAGRA-MUDARRA]

or by

FN'S WITH $C^{1,1}$ NORM ≤ 1.

[WELLS
LEGRUYER]

FIT

MANIFOLDS

WITH

REASONABLE GEOMETRY

TO

DATA

(NOT JUST GRAPHS OF FNS.)

LET'S STOP HERE.

WE'RE DONE!

THANK YOU FOR LISTENING.